

Separable and non-separable data representation for pattern discrimination

Jarosław Adam Mischczak
IITiS PAN
Bałtycka 5, 44100 Gliwice, Poland

University of Cagliari,
Via Is Mirrionis 1, 09123 Cagliari, Italy
`mischczak@iitis.pl`

15/03/2015 (v. 0.35)

Abstract

We provide a complete work-flow, based on the language of quantum information theory, suitable for processing data for the purpose of pattern recognition. The main advantage of the introduced scheme is that it can be easily implemented and applied to process real-world data using modest computation resources. At the same time it can be used to investigate the difference in the pattern recognition resulting from the utilization of the tensor product structure of the space of quantum states. We illustrate this difference by providing a simple example based on the classification of 2D data.

Keywords: pattern discrimination; quantum computing; entanglement of data

1 Introduction

Quantum machine learning aims at merging the methods from quantum information processing and computer engineering to provide new solutions for problems in the areas of pattern recognition and image understanding [6, 11, 10]. From one side the research in this area is focused at applying the methods of quantum information processing for solving problems related to classification and clustering in signal processing. From the other perspective the methods for classification developed in computer engineering are used to find solutions for problems like quantum state discrimination [4, 1, 3, 5], which are tightly connected with the recent developments in quantum cryptography.

The aim of the paper is to present a complete work-flow, based on the mathematical formalism used in quantum information theory, suitable for processing data for the purpose of pattern recognition. The main goal is to introduce the method for describing classical data using entangled quantum states and to investigate the difference between the entangled and the separable representations of states. The presented work-flow enable the processing of real-world data as it can be implemented and executed on standard desktop computers. The main reason why this is possible is the introduction of the quantization procedure which allows the reduction of the dimensionality of the utilized quantum states.

This paper is organized as follows. In Section 2 we provide some notation and initial considerations related to the problem of using quantum states for the purpose of pattern representation and classification. In Section 3 we propose a general framework for encoding data using the

formalism of quantum states and describe the basic classification of the 1D data using this representation. In Section 4 we introduce a formalism for constructing representations of feature vectors and provide some examples of using entangled and separable flavors of the introduced formalism in 2D case. Finally, in Section 5 we provide a summary of the presented work and provide concluding remarks.

2 Preliminaries

We start by introducing the notation from the area of quantum information relevant for the purpose of processing patterns. We also review the problem of pattern classification and introduce the notation used in the paper.

2.1 Kets, tensors and features

In this paper we use the notation of quantum mechanics in which vectors are written using ket symbols, such that

$$|a_1\rangle \quad (1)$$

is a base vector with label a_1 . In the simplest case of two-dimensional space, which can be used to represent a bit on a quantum computer, we have

$$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (2)$$

Accordingly, for N -dimensional space we have

$$|i\rangle \equiv \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{\text{th}} \text{ position}, \quad (3)$$

with $i \in \{0, 1, \dots, N-1\}$. We also assume that ket vectors are normalized.

This notation can be used to express any combination of vectors as

$$a \equiv |a\rangle = \sum_i a_i |i\rangle. \quad (4)$$

Usually we require that $\sum_i |a_i|^2 = 1$.

Let us now assume that we deal with p features, which are measured in the experiment. With each feature we connect a vector space F_i of dimension K_i , necessary to store the possible measurement results.

A pattern consisting of p features can be represented as a tensor product of feature spaces as

$$F_1 \otimes F_1 \otimes \dots \otimes F_p, \quad (5)$$

where \otimes denotes tensor (a.k.a. Kronecker) product [7]. For two vectors $|a\rangle \in F_1$ and $|b\rangle \in F_2$

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{K_1} \end{pmatrix} \quad \text{and} \quad |b\rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{K_2} \end{pmatrix} \quad (6)$$

tensor product is defined as

$$|a\rangle \otimes |b\rangle = \begin{pmatrix} a_1 b_1 \\ \vdots \\ a_1 b_{K_2} \\ \vdots \\ a_{K_1} b_1 \\ \vdots \\ a_{K_1} b_{K_2} \end{pmatrix}, \quad (7)$$

and it is customary to write $|a\rangle \otimes |b\rangle \equiv |a, b\rangle \equiv |ab\rangle$.

One of the features of the above formalism is the seamless notation of the scalar product as

$$a \cdot b \equiv \langle a|b\rangle = \sum_{i,j} a_i b_j \langle i|j\rangle. \quad (8)$$

Elements $\langle i|j\rangle = \delta_{ij}$ can be written as $\langle i|j\rangle$ and they represent scalar products of the base vectors and $\langle a|$ represents the vector dual to $|a\rangle$.

2.2 Pattern recognition

Let us now assume that we are dealing with the problem of the supervised learning. In this situation we have at our disposal a number of representatives associated with labels denoting classes. The classes used to discriminate are constructed using a number of measurement results, each one representing a pattern of a known class [9, 8],

$$C_i = \{c_k^{(i)} : k = 1, \dots, n_i\}, \quad (9)$$

where number of representatives n_i can be different for each class. The set of class elements with the associated labels is referred to as the learning set.

The classification of the unknown pattern x is performed using the score function $g(x, i)$, calculated for the pattern and each class. In the simplest scenario, the score function is calculated as a distance between the unknown pattern and the class representatives. The class representatives $\rho_\mu^{(i)}$ for each class $i = 1, 2, \dots, C$ are built by the appropriate averaging over the class elements.

The unknown pattern is classified according to the greatest value of the score $q^{(i)}(x)$, calculated as the distance between the pattern x and the class representatives $\rho_\mu^{(i)}$, $i = 1, 2, \dots, C$,

$$q^{(i)}(x) = D(\rho^{(i)}, x). \quad (10)$$

3 Pattern representation in quantum formalism

In this section we develop a quantum representation for patterns and classes. We focus on the patterns distributed with the normal distribution, but the provided framework does not depend on this assumption. We start with the simplest case of 1D data and subsequently we consider the possible generalizations of the introduced formalism for the purpose of discrimination patterns described by vectors of features.

3.1 General framework

In order to define a quantum classification procedure one needs to specify the mapping between the space of values and the space of quantum states. This is exactly the counterpart of the quantization function used in machine learning algorithms. However, in this case, this function

is used not only to reduce the dimensionality of the problem, but also to obtain a quantum representation of patterns and classes.

Here we aim at representing data in the form of vectors — pure quantum states — or corresponding projection operators. In principle one can extend the presented formalism to include mixed quantum states, but the particular choice of pure states enables the straightforward utilization of quantum superposition and quantum entanglement.

3.1.1 Distance-based methods

From the quantum information point of view, the most natural formulation of the pattern classification problem is based on the distance minimization.

In distance-based methods we are interested in the distance between the class representative and the pattern we wish to classify [2]. In order to use quantum states for the purpose of pattern classification we use a framework based on the following assumptions:

- (1) Patterns are represented as normalized vectors.
- (2) Classes are represented by the normalized superposition of the representatives.
- (3) Similarity between the patterns is calculated as a scalar product.

The first assumption from the above list requires mapping the classical data on the quantum pure states,

$$x \mapsto |f(x)\rangle. \quad (11)$$

In the following the procedure executing this step will be called the quantum quantization and its goal is twofold. Firstly, it is required to reduce the feature space and generate the quantization symbols. Secondly, it calculates the dimensions of the state space used and maps the quantization symbols onto this space. The quantum quantization step is crucial from the point of view of analyzing the quantum correlations as it effectively defines the structure of the used state space.

For a given quantum quantization procedure and a selected distance measure one obtains a complete procedure for quantum-based classification. In order to classify an unknown pattern x one has to execute the following steps.

Step 1: Perform quantum quantization by mapping class elements

$$c_j^{(i)} \mapsto |f(c_j^{(i)})\rangle.$$

Step 2: Prepare the class representatives $\rho_\mu^{(i)}$ for each class $i = 1, 2, \dots, C$ by the appropriate averaging over the quantum class elements.

Step 3: Calculate the distance between the pattern x and the class representatives $\rho_\mu^{(i)}$, $i = 1, 2, \dots, C$,

$$q^{(i)}(x) = D(\rho^{(i)}, x).$$

Step 4: Classify the pattern according to the greatest value of $q^{(i)}(x)$.

One should note that the class representatives $\rho_\mu^{(i)}$, $i = 1, 2, \dots, C$ can be obtained as a flat superposition of the learning set elements. We will use this method in the next part of the paper.

3.1.2 Nearest neighbors method

The quantum states obtained using the quantum quantization procedure can be used in any scheme for pattern recognition. For example, in the case of k-NN scheme, one needs to perform the following steps.

Step 1: Perform quantum quantization by mapping class elements

$$c_j^{(i)} \mapsto |f(c_j^{(i)})\rangle.$$

Step 2: Calculate the distances $q_j^{(i)}(x) = D(|f(c_j^{(i)})\rangle, x)$ between the class representatives $\rho_\mu^{(i)}$ and the pattern for each class $i = 1, 2, \dots, C$.

Step 3: Classify the pattern according to the voting using values of $q_j^{(i)}(x)$ utilizing k values.

However, in this case the procedure is reduced to the classical one as it does not utilize the superposition of the base vectors. For this reason we restrict our attention to the distance-base methods.

3.2 1D case with flat subspaces

In the following we focus on the features represented as numerical values.

We start by considering the 1D feature vectors and introducing the quantum quantization procedure. In this case features (and class elements) are represented using flat representation in the subspace of the appropriate quantum state space. In this context flat means that patterns are represented by first transforming them into equally distributed intervals and next by representing the intervals as quantum state vectors.

In order to fix the dimension of the quantum state space required to encode the quantization symbols, we will use the data from the learning set. We also require that for the fixed state space of dimension d , vectors $|0\rangle$ and $|d-1\rangle$ will encode all elements smaller and all elements larger than the elements in the learning set. For this reason we required that in the case of two classes

$$d = |\lceil \max(C_1 \cup C_2) \rceil - \lfloor \min(C_1 \cup C_2) \rfloor| + 2. \quad (12)$$

Here $\lceil c \rceil$ is the nearest integer function, sometimes denoted as $\text{nint}(c)$, defined as

$$\lceil c \rceil = \begin{cases} -1^{\text{sgn}(c)} \lceil |c| \rceil, & \text{if } \{c\} < 1/2 \\ -1^{\text{sgn}(c)} \lfloor |c| \rfloor, & \text{if } \{c\} \geq 1/2 \end{cases}, \quad (13)$$

where $\{c\}$ denotes the integer part of c , $\{c\} = ||c| - |c||$. This function is implemented as **rint** function in **math.h** C standard library and as **Round** function in *Mathematica* system.

For a given number c we construct its quantum representation as vector

$$q_{\text{fs}}(c) = \lceil c \rceil - \{\min(C_1 \cup C_2)\} + 1, \quad (14)$$

in d -dimensional space of pure quantum states.

The above procedure can be implemented in *Mathematica* system as

```
qMap[x_, ns1_, ns2_] := Block[{qMin, qMax, qDim},
  qMin = Round[Min[Join[ns1, ns2]]];
  qMax = Round[Max[Join[ns1, ns2]]] + 1;
  qDim = Abs[qMax - qMin] + 2;
  Which[
```

```

x < qMin, Ket[0, qDim],
x > qMax, Ket[qDim - 1, qDim],
True, Ket[Round[x] - qMin + 1, qDim]
]
]

```

Using the above quantization we define for each class its representation as

$$|\mu_i\rangle = \frac{|\tilde{\mu}_i\rangle}{\| |\tilde{\mu}_i\rangle \|}, \quad (15)$$

where $|\tilde{\mu}_i\rangle = \sum_k q_{fs}(c_k^{(i)})$ represents an unnormalized superposition of the elements representing the class.

Finally we introduce quantum flat subspace (QFS) classifier using the score function

$$g_i^{QFS}(c) = |\langle c | \mu_i \rangle|, \quad (16)$$

for $i = 1, 2, \dots, n$ representing classes C_1, C_2, \dots, C_n . Using this we classify pattern c into class i iff

$$g_i^{QFS}(c) \geq g_j^{QFS}(c) \quad (17)$$

for all $i \neq j$.

Example 1 Let us consider two samples of real numbers C_1 and C_2 distributed with $\mathcal{N}(-2, 1)$ and $\mathcal{N}(2, 1)$ respectively.

For the purpose of classification we represent each class as a normalized superposition of its representatives. For example if C_1 is represented by

$$\{-2.24697, -1.17115, -0.882941, -1.9828\} \quad (18)$$

and C_2 by

$$\{0.836746, 1.70144, 3.05605, -0.0344292\} \quad (19)$$

we get quantum representations for both classes as

$$|\mu_1\rangle = \frac{1}{\sqrt{6}} (|0\rangle + |1\rangle + 2|2\rangle) \quad (20)$$

and

$$|\mu_2\rangle = \frac{1}{2} (|3\rangle + |4\rangle + |5\rangle + |6\rangle) \quad (21)$$

respectively for C_1 and C_2 . Both $|\mu_1\rangle$ and $|\mu_2\rangle$ are normalized and each component has an amplitude representing a relative frequency of the appropriate element in the learning set.

One should note that we need to choose the dimension of the subspace encoding the integer part to be large enough for encoding all representatives. For this reason the representation will require higher dimensional state space if the learning set contains more elements. In this example the observations are encoded in the space of dimension $d = 7$.

This classification gives the following output

$$\{(-4, 1), (-3, 1), (-2, 1), (-1, 1), (0, 2), (1, 2), (2, 2), (3, 2), (4, 1)\} \quad (22)$$

where each pair represents a number and the assigned class.

Note that number 4 has been classified as belonging to class 1 due to the lack of the representative corresponding to this number in both classes. For this reason $g_1^{QFS}(4) = g_2^{QFS}(4) = 0$ and the classifier cannot decide about the result solely on the data included in the quantum state. To avoid the inconclusive results we assign the pattern to the arbitrarily chosen class.

4 Higher dimensional generalizations

In this section we provide methods for using the quantum encoded patterns consisting of vectors of features. We introduce two basic schemes for encoding patterns and classes – separable and non-separable. Next, we provide some examples of using these schemes. We also propose a hybrid method which only partially exploits the tensor product structure of multipartite quantum states.

4.1 Quantum representations of feature vectors

Let us use the learning set illustrated in Fig 1 as an initial example. In this case the elements of the class are represented as quantum states $|00\rangle$ and $|11\rangle$. We can use this representation and construct the representation of the class as the vector $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. In this situation we obtain the state which is entangled. However, we can alternatively construct the representation of the class in Fig. 1 component-wise and in this case we obtain a pair of vectors resulting from the quantum averaging of the components, namely $(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle))$. This can be translated into the representation in the tensor product space as $\frac{1}{2}(|0\rangle + |1\rangle + |2\rangle + |3\rangle)$.

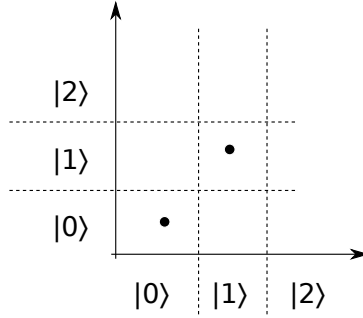


Figure 1: Learning set with two elements from the same class and the possible quantum quantization assigned to each coordinate.

4.2 Separable pattern representation

The most straightforward extension of the 1D example to the case of patterns represented by vectors is obtained by using the 1D classifier independently from each component. In this model the features are encoded into corresponding quantum states and the classification is performed component-wise. As a result one obtains a vector of results, which indicates the classification of the pattern.

Following the example presented above, we introduce the separable data representation as follows.

Definition 1 (Separable data representation) *For a vector of features $x = (x_1, x_2, \dots, x_n)$ the quantum counterpart obtained using the separable data representation is constructed as n -tuple of quantum representations of the components, ie.*

$$|x\rangle = \begin{pmatrix} |x_1\rangle \\ |x_2\rangle \\ \vdots \\ |x_n\rangle \end{pmatrix}. \quad (23)$$

Accordingly, one can introduce the quantum averaging used to represent classes of features as an averaging over the components. This can be expressed as follows.

Definition 2 (Separable class representation) *For a class consisting of n -dimensional features vector, $C_i = \{c_1^{(i)}, c_2^{(i)}, \dots, c_k^{(i)}\}$, the quantum separable representation of the class is obtained as*

$$|\mu_i^S\rangle = \frac{1}{\sqrt{k}} \begin{pmatrix} \sum_{j=1}^k |c_{j,1}^{(i)}\rangle \\ \sum_{j=1}^k |c_{j,2}^{(i)}\rangle \\ \vdots \\ \sum_{j=1}^k |c_{j,n}^{(i)}\rangle \end{pmatrix}, \quad (24)$$

where $c_{j,l}^{(i)}$ denotes the l -th element of the j -th feature vector in the class C_i .

In the simplest case of 2D feature space we have

$$|(v_1, v_2)\rangle = \begin{pmatrix} |v_1\rangle \\ |v_2\rangle \end{pmatrix} \quad (25)$$

and the class representations are constructed as

$$|\mu_k\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sum_{i=1}^2 |v_{1,i}^k\rangle \\ \sum_{i=1}^2 |v_{2,i}^k\rangle \end{pmatrix}. \quad (26)$$

One should note that in the introduced scheme the Dirac notation is overloaded and has to be interpreted according to the type of its argument. Moreover, the quantum quantization procedure required to build the representations of components has to operate on the quantum state space of dimension appropriate for each feature.

4.3 Non-separable pattern representation

One of the main differences between the classical and quantum mechanics is the structure of the space used to describe the composite system. In the classical case such space is constructed using the Cartesian product of the subspaces describing the subsystem, whilst in the quantum case the composite systems are described using the Kronecker product.

In quantum information theory the main phenomenon arising from the use of the tensor product is the entanglement. As the entanglement can be seen as a form of strongly non-classical correlation, we expect that the usage of the tensor product structure allows us encoding and exploiting the correlation between the features. For this reason we introduce the non-separable scheme as follows.

Definition 3 (Non-separable data representation) *For a vector of features $x = (x_1, x_2, \dots, x_n)$ the quantum counterpart obtained using the non-separable data representation is constructed as a tensor product of quantum representations of the components, ie.*

$$|x\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \dots \otimes |x_n\rangle. \quad (27)$$

In the simplest case of 2D feature space we have

$$|(v_1, v_2)\rangle = |v_1\rangle \otimes |v_2\rangle. \quad (28)$$

Definition 4 (Non-separable data representation) *For a class consisting of n -dimensional features vector, $C_i = \{c_1^{(i)}, c_2^{(i)}, \dots, c_k^{(i)}\}$, the quantum non-separable representation of the class is obtained as*

$$|\mu_i^E\rangle = \frac{1}{\sqrt{k}} \sum_k |c_{k,1}^{(i)}\rangle \otimes |c_{k,2}^{(i)}\rangle \otimes \dots \otimes |c_{k,n}^{(i)}\rangle. \quad (29)$$

The main difference between the above formula and the formula in Eq. (24) is that it maps feature vectors onto vector space of dimension $d_1 \times d_2 \times \dots \times d_n$, where $d_i, i = 1, 2, \dots, n$, are dimensions of spaces required to encode features. The separable data representation requires only the space of dimension $\sum_i d_i$.

In the case of 2D feature space the above encoding can be implemented in *Mathematica* as

```
qMuEnt2d[k_, ns_] := Block[{m1, m2, v},
  m1 = qMap[#, ns[1, 1], ns[2, 1]] & /@ ns[k, 1];
  m2 = qMap[#, ns[1, 2], ns[2, 2]] & /@ ns[k, 2];
  v = Plus @@ Flatten[Table[x  $\otimes$  y, {x, m1}, {y, m2}], 1];
  v/Norm[v]
]
```

where function qMap was defined in Section 3.2.

4.4 Example: pattern recognition in 2D

The main advantage of the scheme introduced in this section is that it can be easily implemented and executed for the real-world data. In this section we present a simple example illustrating the difference between the separable and the non-separable representation of data.

In order to execute the classification using the representations introduced above, one has to specify the score function used in the final step. In the case of the non-separable data representation we use the natural choice, namely the overlap between the class representation and the quantum state representing the pattern,

$$g^E(x, i) = |\langle \mu_i^E | x \rangle|. \quad (30)$$

In the case of the separable data representation we will use the score function defined as

$$g^E(x, i) = \sqrt{\sum_k (\langle \mu_{i,k}^S | x_k \rangle)^2}, \quad (31)$$

where $|\mu_{i,k}^S\rangle$ denotes the k -th component of the separable quantum representation defined in Eq. (24).

The above definitions allow the comparison of the efficiency of the pattern classification. For this purpose we have used two 2D Gaussian sources producing the pairs described as $(\mathcal{N}(1, 1), \mathcal{N}(1, 1))$ and $(\mathcal{N}(-2, 1), \mathcal{N}(-2, 1))$.

For the purpose of the presented experiment we have generated a sample of 100 learning sets, each consisting of n elements of both classes, with $n = 2, 3, \dots, 16$. For each learning set we have used two samples of 1000 points generated using the distribution describing the classes. Each of the generated points has been classified using both separable and non-separable quantum data representation. The test has been assessed as positive if the point had been classified to the class corresponding to the appropriate distribution.

The results of the described experiment are presented in Fig. 2. As one can observe the utilization of the non-separable data representation allows better classification of the test data. This is expected as the utilization of the tensor product enables the operations on the larger subspace for the purpose of encoding the quantization symbols.

5 Concluding remarks and further work

In the presented work we have introduced the complete quantum-inspired scheme enabling the processing of real-world data for the purpose of pattern classification. The introduced representations of data are based on the quantization procedure which reduces the dimensionality of

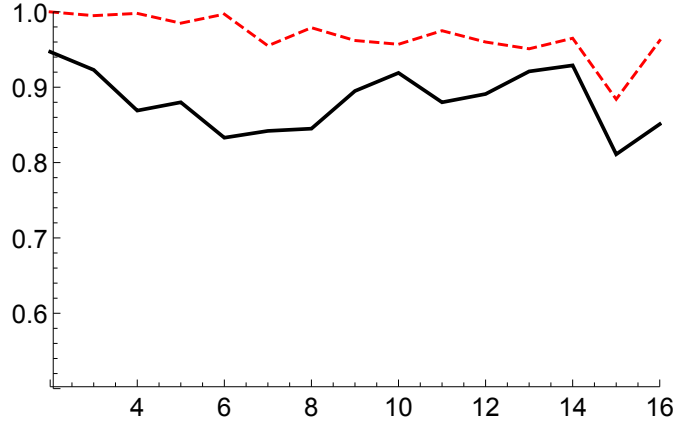


Figure 2: Success rate of the proper classification as the function of the number of elements in the learning set for the separable (black solid line) and non-separable (red dashed line) representation of data.

the quantum state space required to process the data. Thanks to this one can implement and execute the introduced scheme on a standard desktop computer. We have also used the introduced framework to conduct a simple experiment where it is possible to observe the difference between the separable and non-separable representation of data.

One should note that the presented considerations are based on the formalism of quantum mechanics, but do not require a working quantum computer. The utilization of a quantum machine would be beneficial from the security point of view as in this case the information about the class representatives is unknown to the party performing the classification.

The introduced framework is based on a quantization procedure and the introduction of the linear combinations of quantization symbols. As such it can be easily adopted for other problems in machine learning. In particular we plan to apply it to the problem of clustering.

From the quantum information point of view, the introduced framework can be generalized by using a formalism of mixed states in the quantization procedure. One of the advantages of such approach is the possibility of using other distance measures, based on the generalization of distance measures used in classical information theory. Moreover, the use of density matrices can be used for incorporating the information about uncertainty in the measurement results.

Acknowledgments.

This work has been supported by RAS project on: "Modeling the uncertainty: quantum theory and imaging processing", LR 7/8/2007. The author would like to thank P. Gawron, P. Glomb, G. Sergioli and L. Didacci for helpful discussions.

References

- [1] Chefles, A.: Quantum state discrimination. *Contemporary Physics* 41(6) (2000)
- [2] Cunningham, P., Cord, M., Delany, S.: Supervised learning. In: Cord, M., Cunningham, P. (eds.) *Machine Learning Techniques for Multimedia*, pp. 21–49. Cognitive Technologies, Springer Berlin Heidelberg (2008)
- [3] Hayashi, A., Horibe, M., Hashimoto, T.: Quantum pure-state identification. *Phys. Rev. A* 72(5), 052306 (2005)

- [4] Helstrom, C.: Quantum detection and estimation theory. Academic Press (1976)
- [5] Markham, D., Miszczak, J., Puchała, Z., Życzkowski, K.: Quantum state discrimination: A geometric approach. *Phys. Rev. A* 77(4), 042111 (2008), arXiv:0711.4286
- [6] Schuld, M., Sinayskiy, I., Petruccione, F.: An introduction to quantum machine learning. *Contemporary Physics* 0(0), 1–14 (2014), arXiv:1409.3097
- [7] Van Loan, C.: The ubiquitous Kronecker product. *J. Comput. Appl. Math.* 123, 85–100 (2000)
- [8] Webb, A.: Statistical Pattern Recognition. Wiley, 2nd edn. (2002)
- [9] Webb, A., Copsey, K.: Statistical Pattern Recognition. Wiley, 3rd edn. (2011)
- [10] Wiebe, N., Kapoor, A., Svore, K.: Quantum nearest-neighbor algorithms for machine learning. *Quantum Information & Computation* 15(3&4), 0318–0358 (2015).
- [11] Wittek, P.: Quantum Machine Learning: What Quantum Computing Means to Data Mining. Elsevier (2014)